A new subclass of Salagean-type multivalent harmonic function defined by subordination

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Abstract

In this work, we have introduced a new subclass of Salagean-Type multivalent harmonic functions. We give the coefficient bounds, distortion theorems, extreme points, convolution and convex combinations for this subclass of functions. Furthermore, other related results are given as corollaries.

2010 Mathematics Subject Classification. **30C45**. 30C50.
Keywords. Harmonic functions, multivalent functions, subordination, Salagean differential operator.

1 Introduction, main notations and definition

A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain $D \subset \mathbb{C}$ is said to be harmonic in D if u and v are both real harmonic in D. Consider the functions U and V analytic in D so that $u = ReU$ and $v = ImV$. Then the harmonic function f can be expressed by

$$f(z) = h(z) + \overline{g(z)} \qquad (z \in D),$$

where $h = (U + V)/2$ and $g = (U - V)/2$.

In particular, h is called the analytic part and g is called the co-analytic part of f. It is known(see Clunie and Sheil-Small [10]) that the function $f = h + \overline{g}$ is locally univalent and sense-preserving in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \quad and \quad |z| < 1\}$ if and only if $|g'(z)| < |h'(z)| \quad (z \in D)$. The coefficient estimations, distortion theorems, integral expressions, Jacobi estimates and growth condition in geometric properties of covering theorem of the co-analytic part can be obtained by using the analytic part of harmonic functions (see Ahuja [3], Ahuja et al.[4], Aouf and Seoudy [8], Baksa et al. [9], Dixit et al. [11] and the bibliograhy therein). For a fixed positive integer $p \geq 1$, let $SH(p)$ denote the class of all multivalent harmonic functions $f = h + \overline{g}$ which are sense-preserving in the open unit disk $\mathbb{U}$ and of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n + \sum_{n=p}^{\infty} \overline{b_n z^n} \qquad (|b_p| < 1). \tag{1.1}$$

Recent interest in the study of multivalent harmonic function prompted the publication of several articles such as Ahuja and Jahangiri, [1], [2] Jahangiri et al. [17] [18], Yaşar and Yalçın [22], [23],[24] and the references cited therein. We say that a function $f \in SH(p)$ is subordinate to a function $F \in SH(p)$, and write $f(z) \prec F(z)$, if there exists a complex-valued function w which maps $\mathbb{U}$ into oneself with $w(0) = 0$, such that $f(z) = F(w(z)) \quad (z \in \mathbb{U})$. Further, for functions $f_1, f_2 \in SH(p)$ of the forms:

Advanced Studies: Euro-Tbilisi Mathematical Journal Special Issue (10 - 2022), pp. 215–226.

Tbilisi Centre for Mathematical Sciences.
Received by the editors: 11 June 2021.
Accepted for publication: 06 November 2021.

$$f_t(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n + \sum_{n=p}^{\infty} \overline{b_n z^n} \qquad (t = 1, 2).$$

Convolution or Hadamard product of f_1 and f_2 are defined by

$$(f_1 * f_2)(z) = z^p + \sum_{n=p+1}^{\infty} a_{1,n} a_{2,n} z^n + \sum_{n=p}^{\infty} \overline{b_{1,n} b_{2,n} z^n}.$$

Note that the class $SH(p)$ for $p = 1$ was defined and studied by Clunie and Sheil-Small [10]. We propose for the beginning a generalized differential operator as follows.

The differantial operator $D^n \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup 0)$ was introduced by Salagean [19]. On the other hand, for $f = h + \overline{g}$ given by (1.1), (Jahangiri et al.[18]) define the modified Salagean operator D_p^n as

$$D_p^n f(z) = D_p^n h(z) + (-1)^n \overline{D_p^n g(z)} \qquad (p \geq 1, n \in \mathbb{N}_0), \tag{1.2}$$

where

$$D_p^n h(z) = z^p + \sum_{k=p+1}^{\infty} \left[1 + (\frac{k}{p} - 1)\delta\right]^n a_k z^k, \qquad (\delta \geq 0),$$

and

$$D_p^n g(z) = \sum_{k=p}^{\infty} \left[1 + (\frac{k}{p} - 1)\delta\right]^n \overline{b_k z^k}.$$

Associated with the modified Salagean operator D_p^n, we define a subclass $SH_p^{n,\delta}(\alpha, \lambda, A, B)$ of $SH(p)$ consisting of function f of the form (1.1) that satisfy the subordination relation

$$\frac{D_p^{n+1} f(z)}{\lambda D_p^{n+1} f(z) + (1-\lambda) D_p^n f(z)} \prec \frac{p + [pB + (p-\alpha)(A-B)]z}{1 + Bz}, \tag{1.3}$$

where $-B \leq A < B \leq 1$, $0 \leq \alpha < 1$, $0 \leq \lambda < 1$ and $\delta \geq 0$.

Remark 1. For suitably specializing the parameters, the classes $SH_p^{n,\delta}(\alpha, \lambda, A, B)$ reduce to the various subclasses of harmonic univalent functions previously mentioned. Thus, we have the following special cases:

(1) $SH_p^{n,1}(\alpha, 0, A, B) = SH_{p,n}(\alpha, A, B)$ (Çakmak et al. [7])
(2) $SH_1^{0,1}(0, \lambda, A, B) = S_H^*(\alpha, A, B)$ (Çakmak et al. [6])
(3) $SH_1^{n,1}(\alpha, 0, A, B) = SH_n(\alpha, A, B)$ (Altınkaya et al. [5])
(4) $SH_1^{n,1}(0, 0, A, B) = H_n(A, B)$ (Dziok [13])
(5) $SH_1^{1,0}(0, 0, A, B) = S_H^*(A, B)$ (Dziok [12])
(6) $SH_p^{k,1}(0, 0, 2\alpha - 1, 1) = H_p(k+1, k, \alpha)$ (Jahangiri et al.[18])
(7) $SH_p^{1,1}(0, 0, 2\alpha - 1, 1) = K_H(p, \alpha)$ (Ahuja and Jahangiri [1])
(8) $SH_p^{0,1}(0, 0, 2\alpha - 1, 1) = S_H^*(p, \alpha)$ (Ahuja and Jahangiri [1], Jahangiri [15])
(9) $SH_1^{k,1}(0, 0, 2\alpha - 1, 1) = H(k, \alpha)$ (Jahangiri et al. [16])

(10) $SH_1^{1,1}(0,0,2\alpha-1,1) = K_H(\alpha)$ (Jahangiri [15])
(11) $SH_1^{0,1}(0,0,2\alpha-1,1) = S_H^*(\alpha)$ (Jahangiri [15])
(12) $SH_1^{1,1}(0,0,-1,1) = K_H$ (Silverman and Silvia [21])
(13) $SH_1^{0,1}(0,0,-1,1) = S_H^*$ (Silverman [20]).

Further, we define the class $\overline{SH}_p^{n,\delta}(\alpha,\lambda,A,B) \equiv SH_p^{n,\delta}(\alpha,\lambda,A,B) \cap SH_n(p)$, where $SH_n(p)$ 1) denote the class of functions $f = h + \overline{g}$ in $SH(p)$ so that h and g are of the form

$$h(z) = z^p + \sum_{k=p+1}^{\infty} |a_k| z^k$$

and

$$g(z) = (-1)^p \sum_{k=p}^{\infty} |b_k| z^k, \quad |b_p| < 1. \tag{1.4}$$

Using the idea of Dziok and Dziok et al. (see Dziok [12],[13] and Dziok et al. [14]), in this paper, we find necessary and sufficient conditions, distortions bounds, compactness and extreme points for the above defined class $\overline{SH}_p^{n,\delta}(\alpha,\lambda,A,B)$.

2 Main results

In this section we formulate some important results beginning with necessary and sufficient conditions for multivalent harmonic functions in the class $SH_p^{n,\delta}(\alpha,\lambda,A,B)$.

Theorem 2.1. Let $f \in SH(p)$. Then $f \in SH_p^{n,\delta}(\alpha,\lambda,A,B)$ if and if only if

$$D_p^n f(z) * \varphi(z,\zeta) \neq 0 \qquad (\zeta \in \mathbb{C}, \quad |\zeta| = 1, \quad z \in \mathbb{U} - 0),$$

where

$$\varphi(z;\zeta) = z^p \frac{p(1+B\zeta)-(1-\lambda+p\lambda)p+[pB+(p-\alpha)(A-B)]\zeta+(1-p)(1+B\zeta)+(1-\lambda p)p+[pB+(p-\alpha)(A-B)]\zeta z}{(1-z)^2}$$
$$-\overline{z}^p \frac{p(1+B\zeta)-(1-\lambda-p\lambda)p+[pB+(p-\alpha)(A-B)]\zeta+(1-p)(1+B\zeta)-(1-\lambda p)p+[pB+(p-\alpha)(A-B)]\zeta\overline{z}}{(1-\overline{z})^2}.$$

Proof. Let $f \in SH(p)$ be of the form (1.1). The $f \in SH_p^{n,\delta}(\alpha,\lambda,A,B)$ if and only if it satisfies (1.3) or equivalently

$$\frac{D_p^{n+1} f(z)}{\lambda D_p^{n+1} f(z) + (1-\lambda) D_p^n f(z)} \neq \frac{p + [pB + (p-\alpha)(A-B)]\zeta}{1 + B\zeta}, \tag{2.1}$$

where $\zeta \in \mathbb{C}, \quad |\zeta| = 1$ and $z \in \mathbb{U} - 0$. Now as

$$D_p^n f(z) = D_p^n f(z) * \left(\frac{z^p}{1-z} + \frac{\overline{z^p}}{1-\overline{z}} \right)$$

and

$$D_p^{n+1} f(z) = D_p^n f(z) * \left(\frac{z^p[p + (1-p)z]}{(1-z)^2} - \frac{\overline{z^p}[p + (1-p)\overline{z}]}{(1-\overline{z})^2} \right),$$

the inequality (2.1) yields

$$
\begin{aligned}
&(1+B\zeta)D_p^{n+1}f(z)-(p+[pB+(p-\alpha)(A-B)]\zeta)\left(\lambda D_p^{n+1}f(z)+(1-\lambda)D_p^nf(z)\right)\\
&= D_p^nh(z)*\{1+B\zeta-\lambda(p+[pB+(p-\alpha)(A-B)]\zeta)\}\tfrac{z^p[p+(1-p)z]}{(1-z)^2}\\
&\quad -(1-\lambda)\{p+[pB+(p-\alpha)(A-B)]\zeta\}\tfrac{z^p}{1-z}\\
&\quad -(-1)^nD_p^n\overline{g(z)}*\{1+B\zeta-\lambda(p+[pB+(p-\alpha)(A-B)]\zeta)\}\tfrac{\overline{z}^p[p+(1-p)\overline{z}]}{(1-\overline{z})^2}\\
&\quad +(1-\lambda)\{p+[pB+(p-\alpha)(A-B)]\zeta\}\tfrac{\overline{z}^p}{1-\overline{z}}.\\
&= D_p^nf(z)*\varphi(z;\zeta)\neq 0.
\end{aligned}
$$

This completes the proof. Q.E.D.

A sufficient coefficient bound for the class $SH_p^{n,\delta}(\alpha,\lambda,A,B)$ is provided in the following.

Theorem 2.2. Let $f=h+\overline{g}$ be so that h and g are given by(1.1). Then $f\in SH_p^{n,\delta}(\alpha,\lambda,A,B)$, if

$$\sum_{k=p}^{\infty}[1+(\frac{k}{p}-1)\delta]^n\{\varphi_n|a_n|+\psi_n|b_n|\}\leq 2\{p(\lambda-\lambda p+B)+(p^2\lambda+\lambda-1)[pB+(p-\alpha)(A-B)]\} \quad (2.2)$$

with

$$\varphi_n=(1-\lambda p+B)k-(1-\lambda)p-(\lambda pk+1-\lambda)[pB+(p-\alpha)(A-B)] \quad (2.3)$$

and

$$\psi_n=(1-\lambda p+B)k+(1-\lambda)p+(\lambda pk+1-\lambda)[pB+(p-\alpha)(A-B)]. \quad (2.4)$$

Proof. For this purpose, we need to show that if (2.2) holds, then $f\in SH_p^{n,\delta}(\alpha,\lambda,A,B)$ if and only if there exists a complex valued function w; $w(0)=0$, $|w(z)|<1$ $(z\in\mathbb{U})$ such that

$$\frac{D_p^{n+1}f(z)}{\lambda D_p^{n+1}f(z)+(1-\lambda)D_p^nf(z)}=\frac{p+[pB+(p-\alpha)(A-B)]w(z)}{1+Bw(z)}$$

or, alternatively

$$\left|\frac{(1-\lambda p)D_p^{n+1}f(z)-(1-\lambda)pD_p^nf(z)}{\{\lambda p[pB+(p-\alpha)(A-B)]-B\}D_p^{n+1}f(z)+(1-\lambda)[pB+(p-\alpha)(A-B)]D_p^nf(z)}\right|<1.$$

Indeed, letting $|z|=r$ $(0<r<1)$ we obtain

$$
\begin{aligned}
&|(1-\lambda p)D_p^{n+1}f(z)-(1-\lambda)pD_p^nf(z)|\\
&\quad -|\{\lambda p[pB+(p-\alpha)(A-B)]-B\}D_p^{n+1}f(z)+(1-\lambda)[pB+(p-\alpha)(A-B)]D_p^nf(z)|\\
&\qquad =|(1-\lambda p)\{pz^p+\sum_{k=p+1}^{\infty}[1+(\frac{k}{p}-1)\delta]^na_kz^k-(-1)^{n+1}\sum_{k=p}^{\infty}[1+(\frac{k}{p}-1)\delta]^n\overline{b_kz^k}\}
\end{aligned}
$$

$$-(1-\lambda)p\{z^p+\sum_{k=p+1}^{\infty}[1+(\frac{k}{p}-1)\delta]^n a_k z^k+(-1)^{n+1}\sum_{k=p}^{\infty}[1+(\frac{k}{p}-1)\delta]^n\overline{b_k z^k}\}|$$

$$-|\{\lambda p[pB+(p-\alpha)(A-B)]-B\}\{pz^p+\sum_{k=p+1}^{\infty}[1+(\frac{k}{p}-1)\delta]^n a_k z^k-(-1)^{n+1}\sum_{k=p}^{\infty}[1+(\frac{k}{p}-1)\delta]^n\overline{b_k z^k}\}$$

$$+(1-\lambda)[pB+(p-\alpha)(A-B)]\{z^p+\sum_{k=p+1}^{\infty}[1+(\frac{k}{p}-1)\delta]^n a_k z^k+(-1)^{n+1}\sum_{k=p}^{\infty}[1+(\frac{k}{p}-1)\delta]^n\overline{b_k z^k}\}|$$

$$\leq\sum_{k=p+1}^{\infty}[1+(\frac{k}{p}-1)\delta]^n\{(1-\lambda p+B)k-(1-\lambda)p-(\lambda pk+1-\lambda)[pB+(p-\alpha)(A-B)]\}|a_k||z^k|$$

$$+\sum_{k=p}^{\infty}[1+(\frac{k}{p}-1)\delta]^n\{(1-\lambda p+B)k+(1-\lambda)p+(\lambda pk+1-\lambda)[pB+(p-\alpha)(A-B)]\}|b_k||z^k|$$

$$-\{p(\lambda-\lambda p+B)+(p^2\lambda-1+\lambda)[pB+(p-\alpha)(A-B)]\}|z^p|$$

$$=|z^p|\Bigg\{\sum_{k=p+1}^{\infty}[1+(\frac{k}{p}-1)\delta]^n\{(1-\lambda p+B)k-(1-\lambda)p-(\lambda pk+1-\lambda)[pB+(p-\alpha)(A-B)]\}|a_k||z^{k-p}|$$

$$+\sum_{k=p}^{\infty}[1+(\frac{k}{p}-1)\delta]^n\{(1-\lambda p+B)k+(1-\lambda)p+(\lambda pk+1-\lambda)[pB+(p-\alpha)(A-B)]\}|b_k||z^{k-p}|$$

$$-\{p(\lambda-\lambda p+B)+(p^2\lambda-1+\lambda)[pB+(p-\alpha)(A-B)]\}\Bigg\}<0,$$

by (2.2). The harmonic functions

$$f(z)=z^p+\sum_{k=p+1}^{\infty}\left(\frac{p}{p+(k-p)\delta}\right)^n\frac{p(\lambda-\lambda p+B)+(p^2\lambda-1+\lambda)[pB+(p-\alpha)(A-B)]}{\varphi_n}x_n z^n$$

$$+\sum_{k=p}^{\infty}\left(\frac{p}{p+(k-p)\delta}\right)^n\frac{p(\lambda-\lambda p+B)+(p^2\lambda-1+\lambda)[pB+(p-\alpha)(A-B)]}{\psi_n}\overline{y_n z^n}\tag{2.5}$$

where

$$\sum_{k=p+1}^{\infty}|x_n|+\sum_{k=p}^{\infty}|y_n|\quad and\quad |x_p|=1,$$

show that the coefficient bound given by in Theorem 2 is sharp. Since

$$\sum_{n=p}^{\infty}[1+(\frac{k}{p}-1)\delta]^n[\varphi_k|a^k|+\psi_k|b_k|] = \{p(\lambda-\lambda p+B)+(p^2\lambda-1+\lambda)[pB+(p-\alpha)(A-B)]\}\sum_{n=p}^{\infty}(|x_n|+|y_n|)$$

$$= 2\{p(\lambda-\lambda p+B)+(p^2\lambda-1+\lambda)[pB+(p-\alpha)(A-B)]\}$$

the functions of the form (2.5) are in $SH_p^{n,\delta}(\alpha,\lambda,A,B)$. Q.E.D.

Now we show that the bound (2.2) is also for $\overline{SH}_p^{n,\delta}(\alpha,\lambda,A,B)$.

Theorem 2.3. Let $f = h + \overline{g}$ with h and g of the form (1.4). Then $f \in \overline{SH}_p^{n,\delta}(\alpha,\lambda,A,B)$ if and only if the condition (2.2)) holds.

Proof. In view of the Theorem (2.2), it is adequate to show that $f \notin \overline{SH}_p^{n,\delta}(\alpha,\lambda,A,B)$ if condition (2.2) does not hold.We note that a necessary and sufficient condtion for $f = h+\overline{g}$ given by (1.4) to be in class $\overline{SH}_p^{n,\delta}(\alpha,\lambda,A,B)$ is that the coefficient condition (2.2) to be satisfied. Equivalenty, we must have

$$\left|\frac{H(z)}{G(z)}\right| < 1,$$

where

$$H(z) = z^p p(\lambda-\lambda p) + \sum_{k=p+1}^{\infty}\left[1+(\frac{k}{p}-1)\delta\right]^n\{(1-\lambda p)k-(1-\lambda)p\}|a_k|z^k$$

$$+\sum_{k=p}^{\infty}\left[1+(\frac{k}{p}-1)\delta\right]^n\{(1-\lambda p)k+(1-\lambda)p\}|b_k|\overline{z}^k$$

and

$$G(z) = z^p\left\{[pB+(p-\alpha)(A-B)](\lambda p^2-\lambda+1)-pB\right\}$$

$$+\sum_{k=p+1}^{\infty}\left[1+(\frac{k}{p}-1)\delta\right]^n\{(\lambda pk-1+\lambda)[pB+(p-\alpha)(A-B)]+Bk\}|a_k|z^k$$

$$+\sum_{k=p}^{\infty}\left[1+(\frac{k}{p}-1)\delta\right]^n\{(\lambda pk+1-\lambda)[pB+(p-\alpha)(A-B)]+Bk\}|b_k|\overline{z}^k.$$

Therefore, putting $z = r < 1$ we obtain

$$\left\{p(\lambda-\lambda p) + \sum_{k=p+1}^{\infty}\left[1+(\frac{k}{p}-1)\delta\right]^n\{(1-\lambda p)k-(1-\lambda)p\}|a_k|z^{k-p}\right.$$

$$\left.+\sum_{k=p}^{\infty}\left[1+(\frac{k}{p}-1)\delta\right]^n\{(1-\lambda p)k+(1-\lambda)p\}|b_k|\overline{z}^{k-p}\right\}$$

$$\times \left\{[pB+(p-\alpha)(A-B)](\lambda p^2-\lambda+1)-pB\right.$$
$$+\sum_{k=p+1}^{\infty}\left[1+(\frac{k}{p}-1)\delta\right]^n\{(\lambda pk-1+\lambda)[pB+(p-\alpha)(A-B)]+Bk\}|a_k|z^{k-p}$$
$$\left.+\sum_{k=p}^{\infty}\left[1+(\frac{k}{p}-1)\delta\right]^n\{(\lambda pk+1-\lambda)[pB+(p-\alpha)(A-B)]+Bk\}|b_k|\overline{z}^{k-p}\right\}^{-1}$$
$$<1. \tag{2.6}$$

If condition (2.2) does not hold, then condition (2.6) does not hold for r sufficiently close to 1. Hence, there exists $z_0=r_0$ in $(0,1)$ for which the quotient (2.6) is greater than 1 This contradicts the required condition for $f\in\overline{SH}_p^{n,\delta}(\alpha,\lambda,A,B)$, and so the proof is complete.

Q.E.D.

Theorem 2.4. If $f\in\overline{SH}_p^{n,\delta}(\alpha,\lambda,A,B)$, then for $|z|=r<1$

$$|f(z)|\le(1+|b_p|)r^p$$
$$+(\frac{p}{p+\delta})^n\frac{p(\lambda-\lambda p+B)-(\lambda p^2-1+\lambda)[pB+(p-\alpha)(A-B)]-\{p(2-\lambda p-\lambda+B)+(\lambda p^2+1-\lambda)[pB+(p-\alpha)(A-B)]\}|b_p|}{1-\lambda p^2+Bp+B-(\lambda p^2+\lambda p-\lambda+1)[pB+(p-\alpha)(A-B)]}r^{p+1}$$

and

$$|f(z)|\ge(1-|b_p|)r^p$$
$$-(\frac{p}{p+\delta})^n\frac{p(\lambda-\lambda p+B)-(\lambda p^2-1+\lambda)[pB+(p-\alpha)(A-B)]-\{p(2-\lambda p-\lambda+B)+(\lambda p^2+1-\lambda)[pB+(p-\alpha)(A-B)]\}|b_p|}{1-\lambda p^2+Bp+B-(\lambda p^2+\lambda p-\lambda+1)[pB+(p-\alpha)(A-B)]}r^{p+1}.$$

Proof. We only prove the left-hand inequality. The proof for the right-hand is similiar and will be omitted. Let $f\in\overline{SH}_p^{n,\delta}(\alpha,\lambda,A,B)$. Taking the absolute value of f we have

$$|f(z)|\ge(1-|b_p|)r^p-\sum_{n=p+1}^{\infty}(|a_n|+|b_n|)r^n$$

$$\ge(1-|b_p|)$$

$$-\frac{1}{1-\lambda p^2+Bp+B-(\lambda p^2+\lambda p-\lambda+1)[pB+(p-\alpha)(A-B)]}(\frac{p}{p+\delta})^n\sum_{n=p+1}^{\infty}[1+(\frac{k}{p}-1)\delta]^n(\varphi_n|a_n|+\psi_n|b_n|)r^{p+1}$$

$$\ge(1-|b_p|)r^p$$

$$-(\frac{p}{p+\delta})^n\frac{p(\lambda-\lambda p+B)-(\lambda p^2-1+\lambda)[pB+(p-\alpha)(A-B)]-\{p(2-\lambda p-\lambda+B)+(\lambda p^2+1-\lambda)[pB+(p-\alpha)(A-B)]\}|b_p|}{1-\lambda p^2+Bp+B-(\lambda p^2+\lambda p-\lambda+1)[pB+(p-\alpha)(A-B)]}r^{p+1}.$$

Q.E.D.

Corollary 2.5. Let $f = h + \overline{g}$ with h and g of the form (1.4). If $f \in \overline{SH}_p^{n,\delta}(\alpha, \lambda, A, B)$ than

$$\left\{ w : |w| < \frac{\nu(p,n,\alpha,A,B) + \mu(p,n,\alpha,A,B)|b_p|}{1 - \lambda p^2 + Bp + B - (\lambda p^2 + \lambda p - \lambda + 1)[pB + (p-\alpha)(A-B)]} \right\} \subset f(\mathbb{U}),$$

where

$$\nu(p,n,\alpha,A,B) = (p+1)^k \left\{ 1 - \lambda p^2 + Bp + B - (\lambda p^2 + \lambda p - \lambda + 1)[pB + (p-\alpha)(A-B)] \right\}$$
$$-p^k \left\{ p(\lambda - \lambda p + B) - (\lambda p^2 - 1 + \lambda)[pB + (p-\alpha)(A-B)] \right\}$$

and

$$\mu(p,n,\alpha,A,B) = p^k \{ p(2 - \lambda p - \lambda + B) + (\lambda p^2 + 1 - \lambda)[pB + (p-\alpha)(A-B)] \}$$
$$+p^{k+1} \left\{ 1 - \lambda p^2 + Bp + B - (\lambda p^2 + \lambda p - \lambda + 1)[pB + (p-\alpha)(A-B)] \right\}.$$

Theorem 2.6. Set $h_p(z) = z^p$,

$$h_n(z) = z^p - \left[1 + (\frac{k}{p} - 1)\delta \right]^n \frac{p(\lambda - \lambda p + B) + (p^2\lambda - 1 + \lambda)[pB + (p-\alpha)(A-B)]}{\varphi_n} z^n, \ (k = p, p+1, ...)$$

and

$$g_n(z) = z^p + (-1)^k \left[1 + (\frac{k}{p} - 1)\delta \right]^n \frac{p(\lambda - \lambda p + B) + (p^2\lambda - 1 + \lambda)[pB + (p-\alpha)(A-B)]}{\psi_n} \overline{z}^n, (k = p, p+1, ...).$$

Then $f \in \overline{SH}_p^{n,\delta}(\alpha, \lambda, A, B)$ if and only if it can be expressed by

$$f(z) = \sum_{k=p}^{\infty} (x_n h_n(z) + y_n g_n(z)),$$

where $x_n \geq 0$, $\quad y_n \geq 0$ and $\sum_{k=p}^{\infty}(x_n + y_n)$. In particular, the extreme points of $\overline{SH}_p^{n,\delta}(\alpha, \lambda, A, B)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. Suppose that

$$f(z) = \sum_{k=p}^{\infty} (x_n h_n(z) + y_n g_n(z)),$$

$$= \sum_{k=p}^{\infty} (x_n + y_n) z^p - \sum_{k=p+1}^{\infty} \left[1 + (\frac{k}{p} - 1)\delta \right]^n \frac{p(\lambda - \lambda p + B) + (p^2\lambda - 1 + \lambda)[pB + (p-\alpha)(A-B)]}{\varphi_n} x_n z^n$$
$$+ \sum_{k=p}^{\infty} \left[1 + (\frac{k}{p} - 1)\delta \right]^n \frac{p(\lambda - \lambda p + B) + (p^2\lambda - 1 + \lambda)[pB + (p-\alpha)(A-B)]}{\psi_n} y_n \overline{z}^n.$$

Then

$$\sum_{k=p+1}^{\infty}\left[1+(\frac{k}{p}-1)\delta\right]^{n}\varphi_n|a_n|+\sum_{k=p}^{\infty}\left[1+(\frac{k}{p}-1)\delta\right]^{n}\psi_n|b_n|$$

$$=p(\lambda-\lambda p+B)+(p^2\lambda-1+\lambda)[pB+(p-\alpha)(A-B)]\left(\sum_{k=p+1}^{\infty}x_n+\sum_{k=p}^{\infty}y_n\right)$$

$$=p(\lambda-\lambda p+B)+(p^2\lambda-1+\lambda)[pB+(p-\alpha)(A-B)](1-x_p)$$

$$\leq p(\lambda-\lambda p+B)+(p^2\lambda-1+\lambda)[pB+(p-\alpha)(A-B)]$$

and so $f\in\overline{SH}_p^{n,\delta}(\alpha,\lambda,A,B)$. Conversely, if $f\in\overline{SH}_p^{n,\delta}(\alpha,\lambda,A,B)$, than

$$|a_k|\leq\left[1+(\frac{k}{p}-1)\delta\right]^{n}\frac{p(\lambda-\lambda p+B)+(p^2\lambda-1+\lambda)[pB+(p-\alpha)(A-B)]}{\varphi_n}$$

$$|b_k|\leq\left[1+(\frac{k}{p}-1)\delta\right]^{n}\frac{p(\lambda-\lambda p+B)+(p^2\lambda-1+\lambda)[pB+(p-\alpha)(A-B)]}{\psi_n}.$$

Set

$$a_k\leq\left[1+(\frac{k}{p}-1)\delta\right]^{n}\frac{\varphi_n}{p(\lambda-\lambda p+B)+(p^2\lambda-1+\lambda)[pB+(p-\alpha)(A-B)]}|a_n|\quad(k=p{+}1,p{+}2,...)$$

$$b_k\leq\left[1+(\frac{k}{p}-1)\delta\right]^{n}\frac{\psi_n}{p(\lambda-\lambda p+B)+(p^2\lambda-1+\lambda)[pB+(p-\alpha)(A-B)]}|b_n|\quad(k=p,p+1,...).$$

Then note by Theorem (2.2), we have

$$0\leq x_n\leq 1\quad(k=p+1,p+2,...)$$

and

$$0\leq y_n\leq 1\quad(k=p,p+1,...).$$

We define

$$x_p=1-\sum_{k=p+1}^{\infty}x_k-\sum_{k=p}^{\infty}y_k$$

and we note that by Theorem (2.2), $x_p\geq 0$. Consequently, we obtain

$$f(z)=\sum_{k=p}^{\infty}(x_nh_n(z)+y_ng_n(z)).$$

The required outcome is derived. Q.E.D.

Next, we show that $\overline{SH}_p^{n,\delta}(\alpha,\lambda,A,B)$ is closed under convex combinations of its members.

Theorem 2.7. The class $\overline{SH}_p^{n,\delta}(\alpha,\lambda,A,B)$ is closed under convex combination.

Proof. For $m \in \mathbb{N}$, let $f_m \in \overline{SH}_p^{n,\delta}(\alpha,\lambda,A,B)$ be given by

$$f_m(z) = z^p - \sum_{k=p+1}^{\infty} |a_{m,k}| z^k + \sum_{k=p}^{\infty} |b_{m,k}| \overline{z}^k.$$

Then by (2.2), we get

$$\sum_{k=p}^{\infty} \left[1 + (\frac{k}{p} - 1)\delta\right]^n (\varphi_n |a_{m,k}| + \psi |b_{m,k}|) \leq 2\{p(\lambda - \lambda p + B) + (p^2\lambda - 1 + \lambda)[pB + (p-\alpha)(A-B)]\}.$$

For $\sum_{m=1}^{\infty} \mu_m = 1, \quad (0 \leq \mu \leq 1)$, the convex combination of f_m is

$$\sum_{m=1}^{\infty} \mu_m f_m(z) = z^p - \sum_{k=p+1}^{\infty} \left(\sum_{m=1}^{\infty} \mu_m |a_{m,k}|\right) z^k + (-1)^n \sum_{k=p}^{\infty} \left(\sum_{m=1}^{\infty} \mu_m |b_{m,k}|\right) \overline{z}^k.$$

Then on using (2.6), we can write

$$\begin{aligned}
&\sum_{k=p}^{\infty} \left[1 + (\frac{k}{p} - 1)\delta\right]^n \left(\varphi_n (\sum_{m=1}^{\infty} \mu_m |a_{m,k}|) + \psi_k (\sum_{m=1}^{\infty} \mu_m |b_{m,k}|)\right) \\
&= \sum_{m=1}^{\infty} \mu_m \left(\sum_{k=p}^{\infty} \left[1 + (\frac{k}{p} - 1)\delta\right]^n [\varphi_k |a_{m,k}| + \psi_k |b_{m,k}|]\right) \\
&\leq 2\{p(\lambda - \lambda p + B) + (p^2\lambda - 1 + \lambda)[pB + (p-\alpha)(A-B)]\} \sum_{m=1}^{\infty} \mu_m \\
&= 2\{p(\lambda - \lambda p + B) + (p^2\lambda - 1 + \lambda)[pB + (p-\alpha)(A-B)]\}
\end{aligned}$$

and so

$$\sum_{m=1}^{\infty} \mu_m f_m(z) \in \overline{SH}_p^{n,\delta}(\alpha,\lambda,A,B).$$

Q.E.D.

3 Conclusion

In the paper new classes of univalent harmonic functions are introduced. Necessary and sufficient conditions for defined classes of functions are obtained.We also investigated, distortion limits, poblems with extreme points and poperties of closed under convex combination for the newly defined subclasses. For these subclasses, problems like topological properties, integral mean inequalities, and further applications are open problem for the new researchers.

Acknowledgment. This research has been supported by Dicle University Scientific Research Projects Coordination Unit.Project Number:BF.21.001, 2021

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